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Superfield Approach To Exact And Unique Nilpotent Symmetries

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Abstract: In the framework of *usual* superfield approach, we derive the exact local, covariant, continuous and off-shell nilpotent Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetry transformations for the $U(1)$ gauge field (A_μ) and the (anti-)ghost fields $((\bar{C})C)$ of the Lagrangian density of a four $(3+1)$-dimensional QED by exploiting the horizontality condition defined on the six $(4,2)$-dimensional supermanifold. The long-standing problem of the exact derivation of the above nilpotent symmetry transformations for the matter (Dirac) fields $(\bar{\psi}, \psi)$, in the framework of superfield formulation, is resolved by a new restriction on the $(4,2)$-dimensional supermanifold. This new gauge invariant restriction on the supermanifold, due to the *augmented* superfield formalism, owes its origin to the (super) covariant derivatives. The geometrical interpretations for all the above off-shell nilpotent transformations are provided in the framework of *augmented* superfield formalism.

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The current year 2005 has been declared as the "world year of physics" to mark the 100th anniversary of the epoch-making discoveries made by Einstein in his miraculous year 1905. The year 2005 has also been a landmark year for the researchers, working in the realm of Becchi-Rouet-Stora-Tyutin (BRST) formalism, because it has celebrated the 30th birth anniversary of the discovery of BRST symmetries in the context of gauge theories [1,2]. This formalism, during its three decades of existence, has found applications in some of the frontier areas of research like topological field theories [3,4] and string field theories [5].

The key ideas of the BRST formalism have deep connections with the mathematics of differential geometry and (theoretical) physics of gauge theories as well as supersymmetries. One of its intuitive connections is with supersymmetry through the *usual* superfield formulation [6] which provides the geometrical interpretations for the nilpotent ($Q^2_{(a)b} = 0$) and anticommuting ($Q_b Q_{ab} + Q_{ab} Q_b = 0$) (anti-)BRST charges ($Q_{(a)b}$) in a beautiful manner. There exist, however, some long-standing problems in this domain of research which have defied their resolutions during the last 25 years. In our presentation, we shall touch upon one such long-standing problem (connected with the superfield approach to BRST formalism) and provide its resolution by exploiting the importance of *gauge invariance*.

Under the usual superfield approach [6], a D-dimensional Abelian gauge theory (endowed with the first-class constraints in the language of Dirac's prescription [7,8]) is considered on a $(D, 2)$-dimensional supermanifold parameterized by D-number of spacetime (even) co-ordinates x^μ ($\mu = 0, 1, 2, 3....D-1$) and a couple of (odd) Grassmannian variables θ and $\bar\theta$ (with $\theta^2 = \bar\theta^2 = 0, \theta\bar\theta + \bar\theta\theta = 0$). In general, the $(p+1)$-form super curvature $\tilde F^{(p+1)} = \tilde d \tilde A^{(p)}$, constructed from the super exterior derivative $\tilde d$ (with $\tilde d^2 = 0$) and the super p-form connection $\tilde A^{(p)}$ (corresponding to a p-form ($p = 1, 2...$) Abelian gauge theory) is restricted to be flat along the Grassmannian directions of the $(D, 2)$-dimensional supermanifold due to the so-called horizontality condition †. Mathematically, this condition implies $\tilde F^{(p+1)} = F^{(p+1)}$ where $F^{(p+1)} = dA^{(p)}$ is the $(p+1)$-form curvature defined on the ordinary D-dimensional manifold through the ordinary exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) and ordinary p-form Abelian connection $A^{(p)} = \frac{1}{p!}\,[dx^{\mu_1} \wedge dx^{\mu_2} ... \wedge dx^{\mu_p}]\; A_{\mu_1 \mu_2 .....\mu_p}$.

The above horizontality condition on the six $(4, 2)$-dimensional supermanifold leads to the derivation of the nilpotent (anti-)BRST symmetry transformations for the gauge- and (anti-)ghost fields of the (anti-)BRST invariant Lagrangian density of a given four $(3+1)$-dimensional (4D) 1- and 2-form (non-)Abelian gauge theories [6]. However, it does not shed any light on the nilpotent (anti-)BRST symmetry transformations that are associated with the matter (Dirac) fields of the interacting 1-form (non-)Abelian gauge theories where there is a coupling between the gauge field and the matter conserved current, constructed by the Dirac fields. This issue (i.e. the derivation of the nilpotent transformations for matter fields) has been a long-standing problem in the superfield approach to BRST formalism.

In a recent set of papers ‡ [10-13], the usual superfield formalism has been consistently

† Nakanishi and Ojima call it the "soul-flatness" condition [9]. For the 1-form non-Abelian gauge theory, $\tilde F^{(2)} = \tilde d \tilde A^{(1)} + \tilde A^{(1)} \wedge \tilde A^{(1)}$ and $F^{(2)} = dA^{(1)} + A^{(1)} \wedge A^{(1)}$ in the horizontality condition $\tilde F^{(2)} = F^{(2)}$ [6].

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extended by invoking the additional restrictions on the six $(4, 2)$-dimensional supermanifold that are complimentary to the horizontality condition [6]. These additional restrictions on the supermanifold are the equality of (i) the conserved (super) matter current [10,11] (as well as other conserved quantities [11]), and (ii) the gauge invariant quantities owing their origin to the (super) covariant derivatives on the (super) matter fields [12,13].

The former set of restrictions [10,11] lead to the consistent derivation of the nilpotent symmetry transformations for the matter fields. On the other hand, the latter restrictions [12,13] lead to the exact and unique derivation of the nilpotent symmetry transformations for the matter fields. We christen these extended versions of the usual superfield approach to BRST formalism as the augmented superfield formalism. Both types of extensions have their own merits and advantages. Any further (consistent) extension of the usual superfield approach would be a welcome sign for the future of this area of research.

In our presentation, we *first* focus on the strength of the horizontality condition in the exact and unique derivation of the nilpotent symmetry transformations for the gauge and (anti-)ghost fields of a 4D interacting $U(1)$ gauge theory with the Dirac fields. This interacting Abelian system has been taken into consideration *only* for the sake of simplicity. The ideas, proposed in our presentation, can be generalized to a non-Abelian interacting gauge theory in a straightforward manner. Second, we concentrate on the consistent derivation of the nilpotent transformations for the matter (Dirac) fields by exploiting the equality of the conserved matter (super) current on the six $(4, 2)$-dimensional supermanifold. Finally, we obtain the exact and unique nilpotent symmetry transformations for the Dirac fields by exploiting the equality of the gauge invariant quantity on the above supermanifold that owes its origin to the (super) covariant derivatives on the (super) Dirac fields.

Let us begin with the (anti-)BRST invariant Lagrangian density $\mathcal{L}_b$ for the *interacting* four $(3 + 1)$-dimensional $U(1)$ gauge theory in the Feynman gauge [14]

$$\mathcal{L}_b = -\frac{1}{4}\, F^{\mu\nu} F_{\mu\nu} + \bar{\psi}\, (i\gamma^\mu D_\mu - m)\, \psi + B\, (\partial \cdot A) + \frac{1}{2}\, B^2 - i\, \partial_\mu \bar{C} \partial^\mu C, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the antisymmetric field strength tensor for the $U(1)$ Abelian gauge theory that is derived from the 2-form $dA^{(1)} = \frac{1}{2}(dx^\mu \wedge dx^\nu) F_{\mu\nu}$ [§]. As is evident, the latter is constructed by the application of the exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) on the 1-form $A^{(1)} = dx^\mu A_\mu$ which defines the Abelian vector potential A_μ. The gauge-fixing term $(\partial \cdot A)$ is derived through the operation of the co-exterior derivative δ (with $\delta = -*d*, \delta^2 = 0$) on the one-form $A^{(1)}$ (i.e. $\delta A^{(1)} = -*d*A = (\partial \cdot A)$) where $*$ is the Hodge duality operation. The fermionic Dirac fields $(\psi, \bar{\psi})$, with the mass m and charge e, couple to the $U(1)$ gauge field A_μ (i.e. $-e\bar{\psi}\gamma^\mu A_\mu \psi$) through the conserved current $J_\mu = \bar{\psi}\gamma_\mu\psi$.

[§] We adopt here the conventions and notations such that the 4D flat Minkowski metric is: $\eta_{\mu\nu} =$ diag $(+1, -1, -1, -1)$ and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = (\partial_0)^2 - (\partial_i)^2, F_{0i} = E_i = \partial_0 A_i - \partial_i A_0 = F^{i0}, F_{ij} = \epsilon_{ijk} B_k, B_i = (1/2)\epsilon_{ijk} F_{jk}, D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi$ where ϵ_{ijk} is the 3D totally antisymmetric Levi-Civita tensor and electric and magnetic fields are E_i and B_i, respectively. In equation (1), γ's are the usual 4×4 Dirac matrices. Furthermore, the Greek indices: $\mu, \nu, \rho ... = 0, 1, 2, 3$ in (1), correspond to the spacetime directions and Latin indices $i, j, k ... = 1, 2, 3$ stand *only* for the space directions on the 4D spacetime manifold.

The anticommuting ($C\bar{C} + \bar{C}C = 0, C^2 = \bar{C}^2 = 0, C\psi + \psi C = 0$ etc.) (anti-)ghost fields $(\bar{C})C$ are required to maintain the unitarity and "quantum" gauge (i.e. BRST) invariance together at any arbitrary order of perturbation theory for a given physical process ¶. The Nakanishi-Lautrup auxiliary field B is required to linearize the quadratic gauge-fixing term $-\frac{1}{2}(\partial \cdot A)^2$, present in the Lagrangian density (1), in a subtle way.

The above Lagrangian density (1) respects the following off-shell nilpotent ($s_{(a)b}^2 = 0$) and anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) (anti-)BRST ($s_{(a)b}$) ‖ symmetry transformations [14]

$$\begin{array}{lcl} s_b A_\mu &=& \partial_\mu C, \qquad s_b C = 0, \qquad s_b \bar{C} = iB, \qquad s_b \psi = -ieC\psi, \\ s_b \bar{\psi} &=& -ie\bar{\psi}C, \qquad s_b B = 0, \qquad s_b F_{\mu\nu} = 0, \quad s_b(\partial \cdot A) = \Box C, \\ s_{ab} A_\mu &=& \partial_\mu \bar{C}, \qquad s_{ab}\bar{C} = 0, \qquad s_{ab} C = -iB, \qquad s_{ab}\psi = -ie\bar{C}\psi, \\ s_{ab}\bar{\psi} &=& -ie\bar{\psi}\bar{C}, \qquad s_{ab} B = 0, \qquad s_{ab} F_{\mu\nu} = 0, \quad s_{ab}(\partial \cdot A) = \Box \bar{C}. \end{array} \tag{2}$$

The noteworthy points, at this stage, are (i) under the nilpotent (anti-)BRST transformations, it is the kinetic energy term (more precisely $F_{\mu\nu}$ itself) that remains invariant. (ii) The electric and magnetic fields E_i and B_i (that are components of $F_{\mu\nu}$) owe their origin to the operation of cohomological operator d on the one-form $A^{(1)}$. (iii) The symmetry transformations in (2) are generated by the local, conserved and nilpotent charges $Q_{(a)b}$. This statement, for the local generic field $\Sigma(x)$, can be succinctly expressed as

$$s_r\, \Sigma(x) = -i\, [\, \Sigma(x), Q_r\,]_\pm, \qquad\qquad r = b, ab, \tag{3}$$

where $\Sigma(x) = A_\mu(x), C(x), \bar{C}(x), \psi(x), \bar{\psi}(x), B(x)$ and the $(+)-$ signs, as the subscripts on the square bracket, correspond to the (anti-)commutators for the generic local field $\Sigma(x)$ (of the Lagrangian density (1)) being (fermionic)bosonic in nature.

To derive the above anticommuting and nilpotent transformations $s_{(a)b}$ for the bosonic $U(1)$ gauge field A_μ and the fermionic (anti-)ghost fields $(\bar{C})C$, we exploit the *usual* superfield formalism, endowed with the horizontality restriction on a six $(4,2)$-dimensional supermanifold. This supermanifold is parametrized by the superspace coordinates $Z^M = (x^\mu, \theta, \bar{\theta})$ where x^μ $(\mu = 0, 1, 2, 3)$ are a set of four even (bosonic) spacetime coordinates and fermionic θ and $\bar{\theta}$ are a set of two odd (Grassmannian) coordinates. One can define a super 1-form $\tilde{A}^{(1)} = dZ^M \tilde{A}_M$ where the supervector superfield $\tilde{A}_M$ (with $\tilde{A}_M = (B_\mu(x,\theta,\bar{\theta}),\ \mathcal{F}(x,\theta,\bar{\theta}),\ \bar{\mathcal{F}}(x,\theta,\bar{\theta}))$ has the component multiplet superfields $B_\mu, \mathcal{F}, \bar{\mathcal{F}}$. These component superfields can be expanded in terms of the basic fields $(A_\mu, C, \bar{C})$, auxiliary field (B) of the Lagrangian density (1) and some extra secondary fields, as [6]

$$\begin{array}{lcl} B_\mu(x,\theta,\bar{\theta}) &=& A_\mu(x) + \theta\, \bar{R}_\mu(x) + \bar{\theta}\, R_\mu(x) + i\, \theta\, \bar{\theta} S_\mu(x), \\ \mathcal{F}(x,\theta,\bar{\theta}) &=& C(x) + i\, \theta \bar{B}(x) + i\, \bar{\theta}\, \mathcal{B}(x) + i\, \theta\, \bar{\theta}\, s(x), \\ \bar{\mathcal{F}}(x,\theta,\bar{\theta}) &=& \bar{C}(x) + i\, \theta\, \bar{\mathcal{B}}(x) + i\, \bar{\theta}\, B(x) + i\, \theta\, \bar{\theta}\, \bar{s}(x). \end{array} \tag{4}$$

¶The full strength of the (anti-)ghost fields turns up in the discussion of the unitarity and "quantum" gauge (i.e. BRST) invariance for the perturbative computations in the realm of non-Abelian gauge theory where, for each loop diagram of the gauge (gluon) fields corresponding to a physical process, a loop diagram consisting of *only* the (anti-)ghost fields is required to exist as its counterpart (see, e.g., [15] for details).

‖We adopt here the notations and conventions followed in [14]. In fact, in its full glory, a nilpotent ($\delta_B^2 = 0$) BRST transformation δ_B is equivalent to the product of an anticommuting ($\eta C = -C\eta, \eta\bar{C} = -\bar{C}\eta, \eta\psi = -\psi\eta, \eta\bar{\psi} = -\bar{\psi}\eta$ etc.) spacetime independent parameter η and s_b (i.e. $\delta_B = \eta\, s_b$) where $s_b^2 = 0$.

It is straightforward to note that the local fields $R_\mu(x), \bar R_\mu(x), C(x), \bar C(x), s(x), \bar s(x)$ are fermionic (anticommuting) in nature and their number matches with the bosonic (commuting) local fields $A_\mu(x), S_\mu(x), \mathcal{B}(x), \bar{\mathcal{B}}(x), B(x), \bar B(x)$ in (4).

All the secondary fields will be expressed in terms of basic fields $(A_\mu, C, \bar C)$ and the auxiliary field (B) due to the restrictions emerging from the application of horizontality condition. The explicit forms of $\tilde F^{(2)}$ and $F^{(2)}$, in the horizontality restriction, are:

$$\tilde F^{(2)} = F^{(2)}, \;\; \tilde F^{(2)} = \tilde d \tilde A^{(1)} = \tfrac{1}{2}(dZ^M \wedge dZ^N) \tilde F_{MN}, \;\; F^{(2)} = dA^{(1)} = \tfrac{1}{2}(dx^\mu \wedge dx^\nu) F_{\mu\nu}. \quad (5)$$

The super exterior derivative $\tilde d$ and the connection super one-form $\tilde A^{(1)}$, in (5), are

$$\begin{array}{lcl} \tilde d &=& dZ^M\,\partial_M = dx^\mu\,\partial_\mu \;+\; d\theta\,\partial_\theta \;+\; d\bar\theta\,\partial_{\bar\theta}, \\ \tilde A^{(1)} &=& dZ^M\,\tilde A_M = dx^\mu\,B_\mu(x,\theta,\bar\theta) + d\theta\,\bar{\mathcal F}(x,\theta,\bar\theta) + d\bar\theta\,\mathcal F(x,\theta,\bar\theta). \end{array} \quad (6)$$

Mathematically, the above condition (5) implies the "flatness" of all the components of the (anti-)symmetric super curvature tensor $\tilde F_{MN}$ that are directed along the θ and/or $\bar\theta$ directions of the supermanifold. Ultimately, the soul-flatness (horizontality) condition $(\tilde d \tilde A^{(1)} = dA^{(1)})$ of the equation (5) (with $\tilde F^{(2)} = F^{(2)}$), yields **

$$\begin{array}{lcl} R_\mu\,(x) &=& \partial_\mu\,C(x), \qquad \bar R_\mu\,(x) = \partial_\mu\,\bar C(x), \qquad s\,(x) = \bar s\,(x) = 0, \\ S_\mu\,(x) &=& \partial_\mu B\,(x) \qquad B\,(x) + \bar B\,(x) = 0, \qquad \mathcal B\,(x) = \bar{\mathcal B}(x) = 0. \end{array} \quad (7)$$

The insertion of all the above values in expansion (4) leads to the derivation of (anti-)BRST symmetries for the gauge- and (anti-)ghost fields of the theory as ††

$$\begin{array}{lcl} B_\mu^{(h)}\,(x,\theta,\bar\theta) &=& A_\mu(x) + \;\theta\;(s_{ab}A_\mu(x)) + \;\bar\theta\;(s_b A_\mu(x)) + \;\theta\;\bar\theta\;(s_b s_{ab} A_\mu(x)), \\ \mathcal F^{(h)}\,(x,\theta,\bar\theta) &=& C(x) \;+\; \theta\;(s_{ab}C(x)) \;+\; \bar\theta\;(s_b C(x)) \;+\; \theta\;\bar\theta\;(s_b\;s_{ab}C(x)), \\ \bar{\mathcal F}^{(h)}\,(x,\theta,\bar\theta) &=& \bar C(x) \;+\; \theta\;(s_{ab}\bar C(x)) \;+ \bar\theta\;(s_b \bar C(x)) \;+\; \theta\;\bar\theta\;(s_b\;s_{ab}\bar C(x)). \end{array} \quad (8)$$

The above exercise provides the physical interpretation for the (anti-)BRST charges $Q_{(a)b}$ as simply the generators (cf. (3)) of translations (i.e. $\mathrm{Lim}_{\bar\theta\to 0}(\partial/\partial\theta), \mathrm{Lim}_{\theta\to 0}(\partial/\partial\bar\theta)$) along the Grassmannian directions of the supermanifold. It is obvious that now $\tilde d \tilde A^{(1)}_{(h)} = dA^{(1)}$, where $\tilde A^{(1)}_{(h)} = dx^\mu B_\mu^{(h)} + d\theta \bar{\mathcal F}^{(h)} + d\bar\theta \mathcal F^{(h)}$ is the modified version of the 1-form super connection $\tilde A^{(1)}$ (cf. (6)) after the application of the horizontality (soul-flatness) condition.

We now derive the nilpotent symmetry transformations for the matter (Dirac) fields $(\psi, \bar\psi)$ due to the invariance of the conserved matter current of the theory on the supermanifold. We start off with the super expansion of the superfields $(\Psi, \bar\Psi)(x,\theta,\bar\theta))$, corresponding to the ordinary Dirac fields $(\psi, \bar\psi)(x)$ of the Lagrangian density (1), as [10,12]

$$\begin{array}{lcl} \Psi(x,\theta,\bar\theta) &=& \psi(x) + i\,\theta\,\bar b_1(x) + i\,\bar\theta\,b_2(x) + i\,\theta\,\bar\theta\,f(x), \\ \bar\Psi(x,\theta,\bar\theta) &=& \bar\psi(x) + i\,\theta\,\bar b_2(x) + i\,\bar\theta\,b_1(x) + i\,\theta\,\bar\theta\,\bar f(x). \end{array} \quad (9)$$

**In the explicit computation of $\tilde d \tilde A^{(1)}$, we have taken into account $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu, dx^\mu \wedge d\theta = -d\theta \wedge dx^\mu, d\theta \wedge d\bar\theta = d\bar\theta \wedge d\theta$, etc., that emerge from the requirement of the nilpotency of $\tilde d$ (i.e. $\tilde d^2 = 0$).

††For the non-Abelian gauge theory where $F^{(2)} = dA^{(1)} + A^{(1)} \wedge A^{(1)}$, the off-shell nilpotent symmetry transformations for the gauge (i.e. $s_b A_\mu = D_\mu C$) and (anti-)ghost fields (with $s_b C = \frac{1}{2} C \times C$, etc.) were found in a beautiful paper by **Bonora and Tonin** with exactly the same kind of expansion as given in (8) (see, [6] for details). The horizontality condition $(\tilde F^{(2)} = F^{(2)})$ plays an important role in this case, too.

In the limit $(\theta, \bar{\theta}) \to 0$, from the above expansions, we get back the usual Dirac fields $(\psi, \bar{\psi})$ (of the Lagrangian density (1)) and the number of bosonic fields $(b_1, \bar{b}_1, b_2, \bar{b}_2)$ match with the fermionic fields $(\psi, \bar{\psi}, f, \bar{f})$ for the consistency with supersymmetry.

We construct the supercurrent $\tilde{J}_\mu(x, \theta, \bar{\theta})$ with the following general super expansion

$$\tilde{J}_\mu(x,\theta,\bar{\theta}) = \bar{\Psi}(x,\theta,\bar{\theta})\; \gamma_\mu\; \Psi(x,\theta,\bar{\theta}) = J_\mu(x) + \theta\; \bar{K}_\mu(x) + \bar{\theta}\; K_\mu(x) + i\; \theta\; \bar{\theta}\; L_\mu(x), \quad (10)$$

where the above components (i.e. $\bar{K}_\mu, K_\mu, L_\mu, J_\mu$), can be expressed in terms of the components of the basic super expansions (9), as (see, e.g., [10])

$$\begin{array}{l} \bar{K}_\mu(x) = i(\bar{b}_2\gamma_\mu\psi - \bar{\psi}\gamma_\mu\bar{b}_1), \qquad K_\mu(x) = i(b_1\gamma_\mu\psi - \bar{\psi}\gamma_\mu b_2), \\ L_\mu(x) = \bar{f}\gamma_\mu\psi + \bar{\psi}\gamma_\mu f + i(\bar{b}_2\gamma_\mu b_2 - b_1\gamma_\mu\bar{b}_1), \qquad J_\mu(x) = \bar{\psi}\gamma_\mu\psi. \end{array} \quad (11)$$

To be consistent with our earlier observation that the (anti-)BRST transformations $(s_{(a)b})$ are equivalent to the translations along the $(\theta)\bar{\theta}$-directions of the supermanifold, it is straightforward to re-express the expansion in (10) as

$$\tilde{J}_\mu(x,\theta,\bar{\theta}) = J_\mu(x) + \theta\; (s_{ab}J_\mu(x)) + \bar{\theta}\; (s_b J_\mu(x)) + \theta\; \bar{\theta}\; (s_b s_{ab} J_\mu(x)). \quad (12)$$

It can be checked explicitly that, under the (anti-)BRST transformations (2), the conserved current $J_\mu(x)$ remains invariant (i.e. $s_b J_\mu(x) = s_{ab} J_\mu(x) = 0$). Thus, from (11), we have

$$b_1\gamma_\mu\psi = \bar{\psi}\gamma_\mu b_2, \qquad \bar{b}_2\gamma_\mu\psi = \bar{\psi}\gamma_\mu\bar{b}_1, \qquad \bar{f}\gamma_\mu\psi + \bar{\psi}\gamma_\mu f = i(b_1\gamma_\mu\bar{b}_1 - \bar{b}_2\gamma_\mu b_2), \quad (13)$$

as the conditions for $s_{(a)b}J_\mu = 0$. This, ultimately, implies: $K_\mu = L_\mu = \bar{K}_\mu = 0$ in (10).

One of the possible solutions to the above restrictions, present in (13), is [10]

$$\begin{array}{l} b_1 = -e\bar{\psi}C, \qquad b_2 = -eC\psi, \qquad \bar{b}_1 = -e\bar{C}\psi, \qquad \bar{b}_2 = -e\bar{\psi}\bar{C}, \\ f = -ie\; [\; B + e\bar{C}C\;]\; \psi, \qquad \bar{f} = +ie\; \bar{\psi}\; [\; B + eC\bar{C}\;]. \end{array} \quad (14)$$

It is evident that the above expressions are consistent but *not* uniquely determined by the restriction $\tilde{J}_\mu(x,\theta,\bar{\theta}) = J_\mu(x)$ on the supermanifold. However, it should be emphasized that, barring the constant factors, the above solutions are very logical. For instance, for the validity of $b_1\gamma_\mu\psi = \bar{\psi}\gamma_\mu b_2$, the pair of bosonic fields b_1 and b_2 should be proportional to the fermionic fields $\bar{\psi}$ and ψ, respectively. The corresponding equality can be achieved, *only* by bringing in, the (anti-)ghost fields of the theory. There is *no* other possible choice. Thus, we judiciously choose $b_1 \sim \bar{\psi}C$ and $b_2 \sim C\psi$. Rest of the consistent choices of (14) are made on similar line of arguments with appropriate constants i and e thrown in.

The stage is now set for the exact derivation of (14). To this end in mind, we begin with the following *gauge invariant* restriction on the supermanifold [12]

$$\bar{\Psi}(x,\theta,\bar{\theta})\; (\tilde{d} + ie\tilde{A}^{(1)}_{(h)})\; \Psi(x,\theta,\bar{\theta}) = \bar{\psi}(x)\; (d + ieA^{(1)})\; \psi(x), \quad (15)$$

where the superfields Ψ and $\bar{\Psi}$ are from (9). The r.h.s. of the above equation, expressed in terms of the differential dx^μ (as $dx^\mu\bar{\psi}(\partial_\mu + ieA_\mu)\psi$), is obviously a $U(1)$ gauge invariant

quantity. The l.h.s. of the above equation yields the coefficients of the differentials $dx^\mu, d\theta$ and $d\bar\theta$. The analogues of the latter two, as is evident from (15), do not exist on the r.h.s.

It is straightforward to note that the coefficients of $d\theta$, collected from the l.h.s., should be set equal to zero. This requirement leads to the following two independent relationships

$$-i\,\bar\psi\,(\bar b_1 + e\bar C\psi) = 0, \qquad \bar\psi\,(if + e\bar C b_2 - eB\psi) = 0. \tag{16}$$

Similarly, the coefficients of $d\bar\theta$ equal to zero, implies the following relationships [12]

$$-i\,\bar\psi\,(b_2 + eC\psi) = 0, \qquad \bar\psi\,(-if + eC\bar b_1 + eB\psi) = 0. \tag{17}$$

Together, the above two equations, lead to the following results (for $\bar\psi \neq 0$)

$$\bar b_1 = -e\,\bar C\,\psi, \qquad b_2 = -e\,C\,\psi, \qquad f = -ie\,(B + e\bar C C)\,\psi. \tag{18}$$

In fact, out of *exactly* four relations, only *two* in (16) and (17), are independent [12].

We shall focus now on the collection of the coefficients of $dx^\mu, dx^\mu(\theta), dx^\mu(\bar\theta)$ and $dx^\mu(\theta\bar\theta)$. The coefficient of the "pure" dx^μ match from the l.h.s. and r.h.s. Exploiting the inputs from (18), we set equal to zero the coefficient of $dx^\mu(\theta)$ and $dx^\mu(\bar\theta)$. These imply

$$i\,[\,\bar b_2 + e\,\bar\psi\bar C\,]\,[\,D_\mu\psi\,] = 0, \qquad i\,[\,b_1 + e\,\bar\psi C\,]\,[\,D_\mu\psi\,] = 0. \tag{19}$$

The above conditions lead to the exact determination of b_1 and $\bar b_2$ as: $b_1 = -e\bar\psi C, \bar b_2 = -e\bar\psi\bar C$. Here, it will be noted that $D_\mu\psi \neq 0$ for the QED with Dirac fields. Finally, we collect the coefficients of $dx^\mu(\theta\bar\theta)$ and set them equal to zero. This condition implies [12]

$$[\,i\bar f + e\bar\psi\,(B + eC\bar C)\,]\,[\,D_\mu\psi\,] = 0, \tag{20}$$

where we have exploited the inputs from (18) and have inserted the values of b_1 and $\bar b_2$ that were obtained earlier. It is obvious that, for $D_\mu\psi \neq 0$, we obtain the exact value of $\bar f$ as: $\bar f = ie[B + eC\bar C]\bar\psi$. Thus, from the restriction (15), we obtain *exactly* all the values of (14). Insertions of the values of (14) into (9) leads to the following (see, [12] for details)

$$\begin{array}{lcl} \Psi\,(x,\theta,\bar\theta) &=& \psi(x) \;+\; \theta\,(s_{ab}\psi(x)) \;+\; \bar\theta\,(s_b\psi(x)) \;+\; \theta\,\bar\theta\,(s_b\,s_{ab}\psi(x)), \\ \bar\Psi\,(x,\theta,\bar\theta) &=& \bar\psi(x) \;+\; \theta\,(s_{ab}\bar\psi(x)) \;+\bar\theta\,(s_b\bar\psi(x)) \;+\; \theta\,\bar\theta\,(s_b\,s_{ab}\bar\psi(x)). \end{array} \tag{21}$$

This establishes the fact that the nilpotent (anti-)BRST charges $Q_{(a)b}$ are the translations generators $(\mathrm{Lim}_{\bar\theta\to 0}(\partial/\partial\theta))\ \mathrm{Lim}_{\theta\to 0}(\partial/\partial\bar\theta)$ along the $(\theta)\bar\theta$ directions of the supermanifold.

To summarize, the geometrical interpretations for (i) the (anti-)BRST transformations $s_{(a)b}$ and their corresponding generators $Q_{(a)b}$, (ii) the nilpotency property of $s_{(a)b}$ and $Q_{(a)b}$, and (iii) the anticommutativity property of $s_{(a)b}$ and $Q_{(a)b}$, for *all* the fields of QED with Dirac fields, emerge in the framework of augmented superfield formalism. Mathematically, these can be expressed, in an explicit manner, as illustrated below

$$\begin{array}{l} s_b \;\Leftrightarrow\; Q_b \;\Leftrightarrow\; \mathrm{Lim}_{\theta\to 0}\dfrac{\partial}{\partial\bar\theta}, \qquad s_{ab} \;\Leftrightarrow\; Q_{ab} \;\Leftrightarrow\; \mathrm{Lim}_{\bar\theta\to 0}\dfrac{\partial}{\partial\theta}, \\ s^2_{(a)b} = 0 \;\;\Leftrightarrow\;\; Q^2_{(a)b} = 0 \;\;\Leftrightarrow\;\; \Big(\dfrac{\partial}{\partial\theta}\Big)^2 = 0,\; \Big(\dfrac{\partial}{\partial\bar\theta}\Big)^2 = 0, \\ s_b s_{ab} + s_{ab} s_b = 0 \;\Leftrightarrow\; Q_b Q_{ab} + Q_{ab} Q_b = 0 \;\Leftrightarrow\; \dfrac{\partial}{\partial\bar\theta}\dfrac{\partial}{\partial\theta} + \dfrac{\partial}{\partial\theta}\dfrac{\partial}{\partial\bar\theta} = 0. \end{array} \tag{22}$$

The *exact* nilpotent (anti-)BRST symmetries for the matter (Dirac) fields are obtained from the gauge *invariant* restriction (15) on the supermanifold which is different in nature than the gauge *covariant* restriction of the horizontality condition (5) (see, [13] for details).

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